

# Statistical dynamics of a non-Abelian anyonic quantum walk

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We study the single particle dynamics of a mobile non-Abelian anyon hopping around many pinned anyons on a surface. The dynamics is modelled by a discrete time quantum walk and the spatial degree of freedom of the mobile anyon becomes entangled with the fusion degrees of freedom of the collective system. Each quantum trajectory makes a closed braid on the world lines of the particles establishing a direct connection between statistical dynamics and quantum link invariants. We find that asymptotically a mobile Ising anyon becomes so entangled with its environment that its statistical dynamics reduces to a classical random walk with linear dispersion in contrast to particles with Abelian statistics which have quadratic dispersion.

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Anyons are point like particles with more general statistics than bosons or fermions. They were shown to exist in systems where the physics is constrained to two dimensions [1]. Beyond mere possible existence they were found to be a good description for low lying quasi-particle excitations of fractional quantum Hall systems [2, 3] and they exactly describe excitations in various strongly correlated two dimensional spin lattice models [4, 5]. Recently there has been tremendous experimental progress in preparation and control of systems capable of exhibiting topological order [6, 7] with the goal to observe anyonic statistics. This is further motivated by the discovery that braiding some types of non-Abelian anyons can be used for naturally fault tolerant quantum computing [8]. The quantum physics of anyonic systems is very rich but is only beginning to be explored in its own right. For example, there have been investigations of the equilibrium properties of dynamically interacting, but static, non-Abelian anyons in chains [9] and two dimensional lattices [10, 11].

Here we describe a simple model which captures some of the non-equilibrium physics of moving non-Abelian anyons interacting purely due to particle statistics. In Ref. [12] the authors introduced a general protocol for quantum walks with anyons to describe the dynamics of one mobile anyon braiding about other pinned anyons on a surface. It was shown that while for Abelian anyons the dispersion is quadratic as in the usual quantum walk, the non-Abelian walk appears to have richer behavior. Transition from coherent quantum to classical random behavior with linear dispersion occurs when a quantum walk strongly decoheres due to interaction with an environment [13]. Indeed, Ref. [14] found that if one introduces a new coin at every other step, or fewer, in a quantum walk the dispersion is quadratic but if a new coin is introduced every time step then the dispersion is linear. One might expect that statistical interactions of non-Abelian anyons would be sufficient to induce such a transition. We show that this is the case by expressing the statistical dynamics of the mobile anyon as a function of topological invariants of the links of the anyonic worldlines generated during the quantum walk. Specifically, we solve for the asymptotic distribution of the Ising model

non-Abelian anyons,  $\sigma$ , which appear as quasi-particle excitations in the Pfaffian wave function description of the 5/2 filled fractional quantum Hall state [3, 15]. It is the purpose of this work to determine the behaviour of such a system by analytical methods, thus opening the way for modelling complex systems that are of interest to statistical physics [16].

The setup (see Fig. 1a) is a surface with  $n$  vacuum pairs of anyons of topological charge  $\sigma$ , and one member of each pair participates in the dynamics. The other  $n$  anyons are moved out of the way or could be excitations on the boundary [26]. The participating anyons are canonically ordered on the surface with the  $n - 1$  pinned anyons and one mobile *walker* anyon. The walker hops between neighboring sites with spatial index  $s = 1, \dots, n - 1$  and has an additional spin-1/2 degree of freedom (DOF) we dub a coin. The pinned anyons are located in between the sites. The total Hilbert space decomposes as  $\mathcal{H} = \mathcal{H}_{\text{space}} \otimes \mathcal{H}_{\text{coin}} \otimes \mathcal{H}_{\text{fusion}} \simeq \mathbb{C}^{n-1} \otimes \mathbb{C}^2 \otimes \mathbb{C}^D$  which becomes infinite dimensional in the asymptotic limit. Fusion DOFs denumerate the number of distinct measurement outcomes of topological charge when pairs of anyons are fused together [8]. Its size grows like  $D \sim d^n$  where  $d$  is the quantum dimension of the anyons. For  $n$  Ising anyons with total trivial charge, we have  $d = \sqrt{2}$  and  $D = 2^{n/2-1}$ , thus effectively they introduce half a new coin DOF per step of the walk.

The dynamics is modelled by a composition of two discrete unitary steps  $W = TU$  where  $U$  acts on the coin and  $T$  is a conditional braiding operator. It moves the walker to the right or left depending on the coin state:

$$T = \sum_{s=1}^{n-2} |s-1\rangle_{\text{space}} \langle s| \otimes |0\rangle_{\text{coin}} \langle 0| \otimes b_{s-1} + |s+1\rangle_{\text{space}} \langle s| \otimes |1\rangle_{\text{coin}} \langle 1| \otimes b_s$$

where  $\{b_s\}_{s=1}^{n-1}$  is a set of unitary generators of the braid group  $\mathcal{B}_n$ . The particular representation depends on the braiding and recoupling rules for the chosen anyons. Notice that the chirality of the mobile anyonic charge current is fixed counterclockwise by this walk. To make  $T$  unitary we assume periodic boundary conditions ( $|0\rangle_{\text{space}} \equiv |n-1\rangle_{\text{space}}$ ) but will be concerned with walks satisfying  $|n/2| < t$  so that winding around the surface is not an issue.

The system's initial state is  $|\Psi(0)\rangle = |s_0 = \lceil \frac{n}{2} \rceil\rangle_{\text{space}} |0\rangle_{\text{coin}} |\Phi\rangle_{\text{fusion}}$  where  $|\Phi\rangle$  is the vacuum configuration of the  $n$  pairs of anyons with half the members braided to the right. After  $t$  iterations, the state is  $|\Psi(t)\rangle = W^t |\Psi(0)\rangle$  and the reduced state of the spatial DOF of the walker is

$$\rho_{\text{space}}(t) = \text{tr}_{\text{coin}} \text{tr}_{\text{fusion}} |\Psi(t)\rangle \langle \Psi(t)| \\ = \frac{1}{d} \sum_{\vec{a}, \vec{a}'} \text{tr} \mathcal{U}_{\vec{a}\vec{a}'}^t \text{tr} \mathcal{V}_{\vec{a}\vec{a}'}^t |2|\vec{a}'| - t + s_0\rangle \langle 2|\vec{a}'| - t + s_0| \quad (1)$$

where  $\mathcal{U}_{\vec{a}\vec{a}'}^t = (\prod_{r=1}^t P_{a_r} U) |0\rangle_{\text{coin}} \langle 0| (\prod_{r=1}^t P_{a_r'} U^\dagger)$  and  $\mathcal{V}_{\vec{a}\vec{a}'}^t = B_{\vec{a}}^t |\Phi\rangle \langle \Phi| B_{\vec{a}'}^{t\dagger}$ , where  $|\Phi\rangle$  corresponds to the Markov trace state (see Fig. 1b). The coin histories are given by the vectors  $\vec{a}, \vec{a}' \in \{0, 1\}^{\otimes t}$ , and the projectors for each outcome are  $P_{a_j} = |a_j\rangle_{\text{coin}} \langle a_j|$ . The braid word for a given coin history is

$$B_{\vec{a}}^t = \prod_{r=0}^{t-1} b_{s_0 + a_{t-r} + 2(\sum_{j=1}^{t-r-1} a_j) - (t-r)}. \quad (2)$$

For a Hadamard coin flip operation  $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ , and  $\text{tr} \mathcal{U}_{\vec{a}\vec{a}'}^t = \frac{1}{2^t} (-1)^{z(\vec{a}, \vec{a}')}$  where  $z(\vec{a}, \vec{a}') = \sum_{j=1}^{t-1} a_j' a_{j+1} + a_j a_{j+1}$ , is the sum of pairs of consecutive right moves (or 1 outcomes of the coin). The trace over the fusion DOF can be related to the Kauffman bracket of a link, denoted  $\langle L \rangle$  for a link  $L$ . Here we get a link which is the Markov trace over the braid words for the forward and backward time evolution histories as dictated by Eq. 1:

$$L = (B_{\vec{a}'}^{t\dagger} B_{\vec{a}}^t)^{\text{Markov}}. \quad (3)$$

Moreover,

$$\text{tr} \mathcal{V}_{\vec{a}\vec{a}'}^t = \frac{1}{d^{(n-1)2^t}} \langle (B_{\vec{a}'}^{t\dagger} B_{\vec{a}}^t)^{\text{Markov}} \rangle$$

where  $d$  is the quantum dimension of the anyons [12].

Concerning ourselves with the diagonal elements of the spatial probability distribution  $p(s, t) \equiv \langle 2s - t + s_0 | \rho_{\text{space}}(t) | 2s - t + s_0 \rangle$ , we have the constraint that  $|\vec{a}| = |\vec{a}'|$ , i.e. the final position of the walker for the braids  $B_{\vec{a}}, B_{\vec{a}'}$  is the same, and the trace over the coin DOF is non zero only if  $a_t = a_t'$ , i.e. the final step is in the same direction for both paths. The result is:

$$p(s, t) = \frac{1}{d^{(n-1)2^t}} \sum_{\{\vec{a}, \vec{a}'; |\vec{a}|=|\vec{a}'|=s, a_t=a_t'\}} (-1)^{z(\vec{a}, \vec{a}')} \langle L \rangle \quad (4)$$

which for  $d = 1$  and  $\langle L \rangle = 1$  reduces to the usual quantum walk distribution.

The problem of computing the distribution thus reduces to computing statistics of a quantum link invariant. The Kauffman bracket of a link  $\langle L \rangle(A)$  is a Laurent polynomial in the argument  $A$  that is an invariant for framed, unoriented links and is framing dependent. It can be related to the Jones polynomial  $V_L$ , which is an invariant for framed, oriented links but

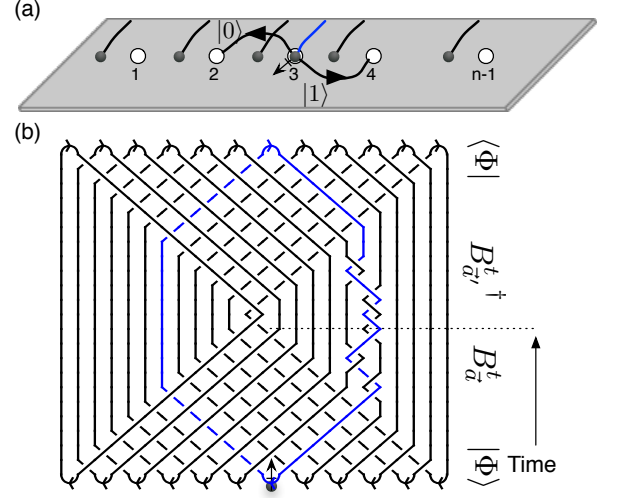


FIG. 1: Anyonic quantum walk in a system with  $n$  vacuum pairs of anyons in state  $|\Phi\rangle$ . (a) Half of each vacuum pair (depicted as connected by a string) participates with one walker anyon carrying a spin braiding around  $n - 1$  pinned anyons. (b) Link representation of the world lines for a Markov closed quantum trajectory that contributes to the spatial distribution  $p(3, 5)$ . Here  $\vec{a} = (1, 0, 0, 1, 1)^T$  and  $\vec{a}' = (0, 1, 1, 0, 1)^T$ . The link shown is proper and has one Borromean ring, i.e. if any of three linked components were cut the others would become disentangled. Here  $\text{arf}(L) = -1$ .

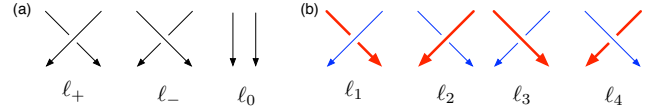


FIG. 2: (a) The writhe of a link is the difference of the number of positive crossing and negative crossings:  $w(L) = \#l_+ - \#l_-$ . (b) The linking number of two components, shown here as thick and thin components, is determined by the relative number of four types of crossings  $lk(L_{\text{thin}}, L_{\text{thick}}) = \frac{\#l_1 + \#l_2 - \#l_3 - \#l_4}{2} = \#l_1 - \#l_4 = \#l_2 - \#l_3$ .

which is framing independent. The relation is achieved by introducing an orientation to each component in  $L$  and suitably normalizing:

$$\langle L \rangle(A)|_{A \rightarrow q^{-1/4}} = (-q^{3/4})^{w(L)} V_L(q)$$

in such a way that the framing dependence of the Kauffman bracket is exactly canceled by the multiplicative factor involving the writhe  $w(L)$  of  $L$  (defined in Fig. 2a). The Jones polynomial of a link  $L$  with variable  $q = e^{i2\pi/(k+2)}$  was shown by Witten [17] to be equal to the expectation value of the product of the path ordered Wilson loops along the components of the links of  $L$  in  $SU(2)_k$  Chern-Simons theory.

At the special value  $q = i(k = 2)$ , where the links represent braiding of anyons in the Ising model, the Jones polynomial of a link can be related to a simpler knot invariant known as

the  $\text{arf}$  invariant [18]. Specifically,

$$V_L(i) = \begin{cases} \sqrt{2}^{\#(L)-1} (-1)^{\text{arf}(L)} & \text{if } L \text{ proper} \\ 0 & \text{if } L \text{ not proper} \end{cases}$$

where the number of components  $\#(L) = n$  here. An oriented link is proper if each component  $L_k$  evenly links the union of other components, i.e.  $\sum_{j \neq k} lk(L_j, L_k) = 0 \pmod{2} \forall j$ . The linking number of two components is defined in Fig. 2b and can be computed in polynomial time. The advantage of this expression is that  $\text{arf}(L) \in \{0, 1\}$  of a link can be computed in polynomial time in the crossing number of a braid presentation. Hence, unlike the generic case, the Jones polynomials at value  $q = i$  can be evaluated in polynomial time [19]. However, the number of links contributing to the weight  $p(s, t)$  is  $\binom{t-1}{s}^2 + \binom{t-1}{s-1}^2$  which is exponential in  $t$  so an efficient computation is not a priori available.

In the special case the link  $L$  has all pairwise linking numbers even, i.e. it is totally proper, then there is a three local formula for the  $\text{arf}$  invariant:

$$\text{arf}(L) = \sum_i c_1(L_i) + \sum_{i < j} c_2(L_i, L_j) + \sum_{i < j < k} c_3(L_i, L_j, L_k) \pmod{2} \quad (5)$$

where  $c_s(\Gamma)$  is the coefficient of  $z^{s+1}$  in the Alexander-Conway polynomial of the  $s$  component sublink  $\Gamma$  [18, 20].

There is structure to the links in the anyonic quantum walk trajectories that simplifies Eq. 5. First, if  $L$  is proper than it is totally proper. This is due to the following: if we consider the set of all the sums  $S = \{\sum_{k \neq j} lk(L_j, L_k)\}_{j=1}^n$ , this is equal to  $S = \{lk(L_j, L_w)\}_{j \neq w} \sqcup \{\sum_{k \neq w} lk(L_w, L_k)\}$  where  $L_w$  is the walker's component. The condition of  $L$  being proper is that every member of  $S$  is an even integer. Now, the total linking number is  $\sum_{i < j} lk(L_i, L_j) = \sum_{k \neq w} lk(L_w, L_k) = 0$ . This quantity is zero since for the forward half of the braid, i.e.  $B_{\vec{a}}^t$ , all contributions to links are positive (counterclockwise braiding) and for the latter half,  $B_{\vec{a}'}^t$ , there is the same number of clockwise braids all of whom contribute with a negative sign, so  $S = \{lk(L_j, L_w)\}_{j \neq w} \sqcup \{0\}$ . But the condition that  $L$  is totally proper is precisely that every member of the set  $\{lk(L_j, L_w)\}_{j \neq w}$  is an even integer. Hence if  $L$  is proper then it is totally proper. Second, the writhe of any link  $L$  is zero since the total linking number is zero and the braids act on components all with the same orientation (see Fig. 1b).

With regard to the terms in Eq. (5), note that due to causality of the worldlines there is no self linking, i.e.  $c_1 = 0$ . The pairwise contribution to the  $\text{arf}$  is also zero since the sum of  $c_2(L_j, L_k)$  is even as we now show. It is only necessary to consider links involving the walker  $L_w$  and every other component  $L_j$  since the non-walker components are not directly linked. Such a link is the braid closure of a braid in the two component braid group  $\mathcal{B}_2$  with one generator  $b$  so that link can be written,  $(L_w, L_j) = (b^m)^{\text{Markov}}$  where  $m = 2 \times lk(L_w, L_j)$ . To compute the two point invariant we use the defining Skein relation for the Alexander-Conway polynomial

$$\nabla_{\ell_+} - \nabla_{\ell_-} = z \nabla_{\ell_0} \quad (6)$$

with the notation defined in Fig. 2a. Moreover,  $\nabla_O = 1$  which states that if one of the components is unlinked from the others then it can be removed with a multiplicative factor of 1. The polynomial for any link  $L$  can be written as  $\nabla_L(z) = \sum_{i=0}^{\infty} a_i z^i$ . For our pairwise component links  $(L_w, L_j)$  one can solve for the polynomial by recursion:

$$\nabla_{(L_w, L_j)} = f_{|m|}(z) + f_{|m|-1}(z),$$

where

$$f_m(z) = \frac{1}{2^{m+1} \sqrt{z^2+4}} \left( (z - \sqrt{z^2+4}) ((z + \sqrt{z^2+4})^m - (z - \sqrt{z^2+4})^m) \right).$$

From the coefficient of the cubic term of  $\nabla_{(L_w, L_j)}$  we obtain

$$c_2(L_w, L_j) = lk(L_w, L_j)(lk(L_w, L_j)^2 - 1)/6.$$

Note that the linking numbers are always even,  $lk(L_w, L_j) = 2n_j$  with  $n_j$  an integer, so the sum can be written as

$$\begin{aligned} \sum_{j < k} c_2(L_j, L_k) &= \frac{8}{6} \sum_{j \neq w} n_j^3 - \frac{8}{6} \sum_{j \neq w} n_j \\ &= 8 \sum_{j \neq w} \binom{n_j+1}{3} \in 2\mathbb{N}. \end{aligned}$$

Here we used the facts that the sum of the linking numbers is zero, so that  $\sum_{j \neq w} n_j = 0$ , and the binomial coefficient is always an integer. Finally, the triple component invariant  $c_3(L_r, L_s, L_t)$  is known as the Milnor invariant. It counts the number of Borromean rings in three component sublinks [21] and has nontrivial contribution to the  $\text{arf}$  of our links.

This analysis allows us to express the probability distribution in terms of simple properties of the anyonic walk:

$$p(s, t) = \sum_{\substack{\{\vec{a}, \vec{a}' \in \{0, 1\}^{\otimes t}; \\ |\vec{a}| = |\vec{a}'| = s, a_t = a'_t}} \left\{ \begin{array}{ll} \frac{(-1)^{z(\vec{a}, \vec{a}') + \tau(\vec{a}, \vec{a}')}}{2^t} & L \text{ proper} \\ 0 & L \text{ not proper} \end{array} \right. \quad (7)$$

where the link  $L$  is defined in Eq. (3) and the sum of Milnor invariants over sublinks  $(L_r, L_s, L_t)$  of  $L$  is  $\tau(\vec{a}, \vec{a}') = \sum_{r < s < t} c_3(L_r, L_s, L_t)$ . When all links are proper and  $\tau(\vec{a}, \vec{a}') = 0$  is even then  $p(s, t) \rightarrow p_{\text{QW}}(s, t)$  the quantum walk distribution and when all non-mirror paths (i.e.  $\vec{a} \neq \vec{a}'$ ) are non proper then  $p(s, t) \rightarrow p_{\text{RW}}(s, t)$  the classical random walk distribution.

To probe the behaviour of  $p(s, t)$  we initially perform exact numerical simulations. Note that  $z(\vec{a}, \vec{a}')$  is a sum of local characteristics of the links, while  $\tau(\vec{a}, \vec{a}')$  is a sum of non-local ones. To compute quantities which involve pairs or triples of components, we make use of the braid word representations of the links Eqs. (2,3) determined by the coin histories  $(\vec{a}, \vec{a}')$ . The linking number  $lk(L_w, L_j) = (\#b_j - \#b_j^\dagger)/2$  when the walker's component is to the right of  $j$  ( $w > j$ ) and  $lk(L_w, L_j) = (\#b_j - \#b_j^\dagger)/2$  otherwise. If  $lk(L_w, L_j)$  is an odd integer for any component  $L_j$  then  $L = (B_{\vec{a}'}^t B_{\vec{a}}^t)^{\text{Markov}}$  is not proper. The Milnor invariant  $c_3(L_r, L_w, L_s)$  where the components are ordered in position with  $r < w < s$ , can be

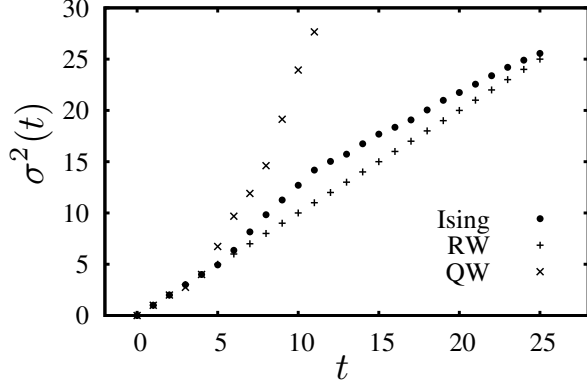


FIG. 3: The time evolution of the variance of the spatial distribution  $p(s,t)$  (Eq. 7) for the Ising anyonic walk and the corresponding classical and quantum walk evolutions with the same initial state.

computed by writing down the braid word  $(B_{a'}^t B_a^t)^{\text{Markov}}$  and deleting all instances of braid word generators not in the set  $\{b_r, b_{s-1}, b_r^\dagger, b_{s-1}^\dagger\}$ . The remaining braid word  $B(L_r, L_w, L_s)$  involves just two generators and their adjoints. We can think of this as a braid word on the three strands braid group. For the spin-1/2 irrep of the Ising model, the representation of these generators is:

$$b_r = -e^{-i\pi/8} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad b_{s-1} = -\frac{e^{i\pi/8}}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}.$$

Note that  $b_r^2 = e^{-i\pi/4} \sigma^z$  and  $b_{s-1}^2 = e^{-i\pi/4} \sigma^x$  where  $\sigma^{x,z}$  are the Pauli operators. For proper links the product of this representation of the braid generators in  $B(L_r, L_w, L_s)$  will be  $\pm 1$  with the sign carrying the value  $(-1)^{c_3(L_r, L_w, L_s)}$ . The Milnor invariant for the other two orderings of components is calculated analogously. We have calculated the distribution for walks up to  $t = 25$ . The variance  $\sigma^2(t) = \langle s^2 \rangle - \langle s \rangle^2$  where the expectation value is  $\langle O(s) \rangle \equiv \sum_s O(s) p(s,t)$ , is plotted in Fig. 3 and it quickly approaches the linear random walk variance. Using the total variation distance between two distributions  $p(s,t)$  and  $f(s,t)$  defined  $\Delta(p, f) \equiv \frac{1}{2} \sum_s |p(s,t) - f(s,t)|$ , at  $t = 25$  we find  $\Delta(p, p_{\text{QW}}) = 0.34$  while  $\Delta(p, p_{\text{RW}}) = 0.04$ .

We now show that asymptotically the Ising anyonic walk behaves classically. The essential reason is the rapidly decreasing density of proper links in the regime where the quantum walk distribution has dominant support. In order to upper bound the variance we can assume that  $\tau(\vec{a}, \vec{a}')$  of all the proper links are even and that there is no correlation between being proper and  $z(\vec{a}, \vec{a}')$  for non mirror paths. Calling the resulting distribution  $\tilde{p}(s,t)$  we have

$$\tilde{p}(s,t) = p(s,t)_{\text{RW}} + p_{\text{prop}}(s,t)[p(s,t)_{\text{QW}} - p(s,t)_{\text{RW}}], \quad (8)$$

where  $p_{\text{prop}}(s,t)$  is the density of proper links for non-mirror paths ( $\vec{a} \neq \vec{a}'$ ). Since the walker's speed is constant, the maximum possible variance is quadratic, achieved up to a constant

less than one by the quantum walk [22], so this choice of distribution can only make the estimate of the variance of the anyonic walker larger, i.e.  $\tilde{\sigma}^2(t) \geq \sigma^2(t)$ .

In Ref. [22] it was shown that the distribution of the quantum walk with the same Hadamard coin flip and initial state as occurs here can be very well approximated asymptotically by the function  $p'_{\text{QW}}(\alpha, t) d\alpha = \frac{(1-\alpha)d\alpha}{\pi(1-\alpha^2)\sqrt{1-2\alpha^2}}$  where  $\alpha = (s_0 - s)/t$  and is restricted to the interval  $[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$ . Outside this interval the distribution falls off exponentially with  $t$  as does  $p_{\text{QW}}(s,t)$ . Restricting to this interval and using the fact that a proper link must have all linking numbers even, it is shown in the Appendix that the density of non-mirror proper links is  $p_{\text{prop}}(s,t) < C/t^2$  for some constant  $C$  independent of  $s, t$ . The position moments with respect to  $\tilde{p}(s,t)$  are

$$\begin{aligned} \langle (s - s_0) \rangle &\approx \int_{-1/\sqrt{2}}^{1/\sqrt{2}} p_{\text{prop}}(s,t) t \alpha p'_{\text{QW}}(\alpha, t) d\alpha \\ &< C(1 - 1/\sqrt{2})/t, \\ \langle (s - s_0)^2 \rangle &\approx t(1 - C/t^2) + \int_{-1/\sqrt{2}}^{1/\sqrt{2}} p_{\text{prop}}(s,t) t^2 \alpha^2 p'_{\text{QW}}(\alpha, t) d\alpha \\ &< t(1 - C/t^2) + (1 - 1/\sqrt{2})C, \end{aligned}$$

Thus  $\tilde{\sigma}^2(t) < t + O(1)$ .

Similarly, to obtain a lower bound for the dispersion we can assume a distribution of the form of Eq. 8 but chose a probability distribution  $f(s,t)$  with minimum variance to replace  $p_{\text{QW}}(s,t)$ . Picking  $f(s,t) = \delta_{s,s_0}$  (zero variance) and calling the resulting distribution  $\tilde{p}(s,t)$  then  $\tilde{\sigma}^2(t) > t(1 - C/t^2)$ . By the inequalities  $\tilde{\sigma}^2(t) \leq \sigma^2(t) \leq \tilde{\sigma}^2(t)$  we find

$$\lim_{t \rightarrow \infty} \frac{\sigma^2(t)}{t} = 1, \quad (9)$$

i.e. asymptotically the Ising anyonic walk has linear dispersion with coefficient 1.

In conclusion, we have studied the dynamical behavior of a mobile non-Abelian anyon which becomes entangled with its environment purely by statistical interactions. We find that for the case of Ising anyons the decoherence is strong enough to completely wash out the quantum mechanical interferences and reduce the dynamics to a classical stochastic process. It would be of interest to extend this analysis to anyons which are spin-1/2 irreps for  $SU(2)_{k>2}$  models since it is known that for  $k > 2, k \neq 4$ , the braiding evolutions densely span the fusion space while for  $k = 2$  (Ising anyons) and  $k = 4$  they do not [23]. It has been shown that for a  $t$  step quantum walk subject to decoherence in its position at a rate  $p_{\text{meas}} > C'/t$  for some constant  $C'$ , the evolution approaches classical behavior [24]. We might then conjecture that since the braiding generators that entangle new fusion degrees of freedom can be translated into a measurement error  $p_{\text{meas}} \sim 1/k^2$  (for  $k \gg 1$ ), then for  $t > k^2$  the walk would behave classically. In the limit  $k \rightarrow \infty$  the particles are fermions and we recover quantum walk behavior with quadratic dispersion. In general a mobile anyons experiences an environmental coupling that is highly non-local and non-Markovian and its behavior sheds light on the statistical dynamics of more complex systems.



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#### Appendix: Upper bound for $p_{\text{prop}}(s, t)$

We will show that  $p_{\text{prop}}(s_0, t)$ , the probability of a randomly chosen link  $L$  to be proper (mirror paths excluded), is bounded from above by a function  $\frac{C}{t^2}$  under certain assumptions. We first consider the number of links with even linking numbers and derive  $p_{\text{prop}}(s_0, t)$  as the proportion of these links with respect to all links. The result holds for paths which end up on the initial site but numerical results suggest that  $p_{\text{prop}}(s, t) < p_{\text{prop}}(s_0, t)$  if  $s \neq s_0$ , provided  $s$  is not too close to the boundaries at  $s_0 \pm t$ . In particular, it is true inside the domain  $\frac{s_0 - s}{t} = [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$ .

Suppose a path  $(\vec{a}, \vec{a}')$  leading after  $t$  steps to the position  $s_0$  links with link components  $L_j$ ,  $j \in \{l, \dots, r\}$  (Fig. 4). For the corresponding link  $L$  to be proper, all the linking

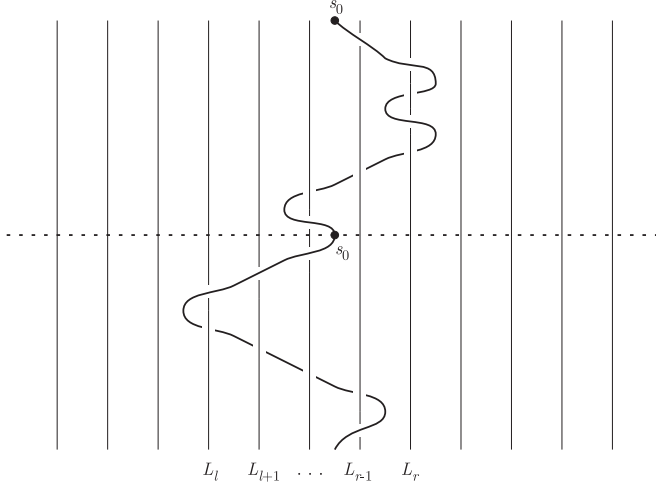


FIG. 4: A generic path reaching after  $t$  steps (here  $t = 8$ ) the position  $s_0$ . The path braids with all the link components  $L_l, \dots, L_r$ . We say that its width  $w$  is  $r - l + 1$  (here  $w = 5$ ).

numbers  $lk(L_w, L_l), \dots, lk(L_w, L_r)$  must be even. Let  $p_e(t, j)$  be the probability that  $lk(L_w, L_j)$  is even. We assume that for large enough (fixed)  $t$ , the probabilities  $p_e(t, j)$  can be treated as independent. This is justified inside the interval  $\frac{s_0 - s}{t} = [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$ , where the number of paths that contribute to the anyonic walk density at each point  $s$  is exponential in  $t$ . The linking number of the walker with any particular component can change even/odd parity by a simple deformation and within the typical width (i.e. number of components touched by the walker) of the path there are an exponential number of

such deformations hence the linking numbers of the walker with those components are well approximated as independent quantities.

Then we can write the probability for the link  $L$  to be proper as

$$p_{\text{prop}}(s_0, t) = p_e(t, l) \cdot \dots \cdot p_e(t, r). \quad (10)$$

If we denote

$$\rho \equiv \max_j p_e(t, j), \quad (11)$$

where  $j$  runs over all link components  $L_j$  that braid with some  $t$  step long path leading to  $s_0$ , we can estimate from above

$$p_{\text{prop}}(s_0, t) \leq \rho^w, \quad (12)$$

where  $w = r - l + 1$  we will call the *width* of the link.

The paths  $(\vec{a}, \vec{a}')$  relevant for a walk of  $t$  steps must satisfy  $a_t = a'_t$ . Let us consider the paths leading to  $s_0$  (we presume  $t$  is even), for which  $a_t = a'_t = 1$ . It is useful to depict  $\vec{a}$  as a lattice path on a  $(n-1) \times n$  lattice ( $n = \frac{t}{2}$ ) with allowed steps  $\uparrow$  and  $\rightarrow$  (Fig. 5). The number of all such paths is

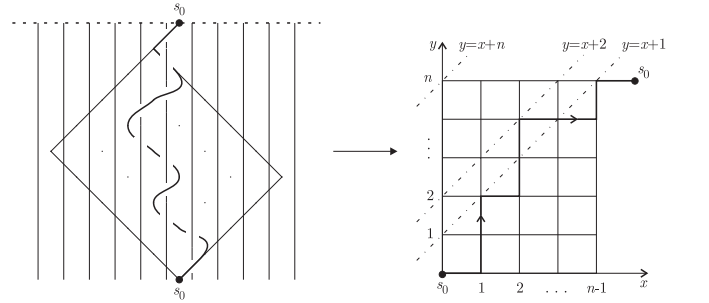


FIG. 5: Any half-path (i.e. either "bra" or "ket" history of the walker) can be thought of as being realized on a two dimensional lattice, where only the steps  $\uparrow$  and  $\rightarrow$  are allowed. If we fix the number of steps  $t$  (presume  $t$  even), the position reached after  $t$  steps  $s = s_0$  and the last coin outcome  $a_t = 1$ , we can draw the possible half-paths on a lattice  $(n-1) \times n$ , where  $n = \frac{t}{2}$ . The lattice paths that touch the diagonal  $y = x + w$  ( $1 \leq w \leq n$ ), but don't touch  $y = x + w + 1$ , braid with at least  $w$  link components. "At least", because they certainly braid with the  $w$  components to the left of  $s_0$ , but they might be (and typically are) spread to the right of  $s_0$  as well.

$$\binom{2n-1}{n-1} \equiv \#(all). \quad (13)$$

The number of lattice paths that touch the diagonal  $y = x + w$ ,  $w \in \{1, \dots, n\}$  is in our case of  $(n-1) \times n$  lattice

$$\binom{2n-1}{n-w} \equiv \#(w). \quad (14)$$

To derive this result we notice that  $\#(1) = \#(all)$  and for  $w \in \{2, \dots, n\}$  we use the *André reflection principle*, described, for example, in [25].

Consider the set  $\mathcal{P}_w$  of lattice paths that touch the diagonal  $y = x + w$  but do not touch the further diagonal  $y = x + w + 1$ . The sets  $\mathcal{P}_w$  are for distinct  $w \in \{1, \dots, n\}$  obviously disjoint. The number of paths in  $\mathcal{P}_w$  is

$$|\mathcal{P}_w| = \#(w) - \#(w+1) = \binom{2n-1}{n-w} - \binom{2n-1}{n-w-1}. \quad (15)$$

We use the convention that  $\binom{m}{k} = 0$  if  $k < 0$ . Furthermore, the union  $\bigcup_{w=1}^n \mathcal{P}_w$  comprises all the paths on  $n-1$  by  $n$  lattice, which follows from

$$\sum_{w=1}^n |\mathcal{P}_w| = \sum_{w=1}^n \left( \binom{2n-1}{n-w} - \binom{2n-1}{n-w-1} \right) = \binom{2n-1}{n-1} = \#(all). \quad (16)$$

A path  $(\vec{a}, \vec{a}')$  consists of two lattice paths,  $\vec{a}$  and  $\vec{a}'$ , where  $\vec{a} \in \mathcal{P}_w$  and  $\vec{a}' \in \mathcal{P}_{w'}$  for some  $w, w' \in \{1, \dots, n\}$ , i.e.  $(\vec{a}, \vec{a}') \in \mathcal{P}_w \times \mathcal{P}_{w'}$ . Let's realize that the width of the path  $(\vec{a}, \vec{a}')$  is at least  $\max\{w, w'\}$  (see again Fig.4 and Fig.5). The probability that the path  $(\vec{a}, \vec{a}')$  is proper can thus be estimated from above (using (12)) as

$$p_{\text{prop}}(s_0, t) \leq \rho^{\max\{w, w'\}}. \quad (17)$$

Finally, the set of all non-mirror paths with the last step "1" reaching after  $t$  steps the starting position  $s_0$  can be expressed in terms of  $\mathcal{P}_w$ 's as shown in Fig.6.

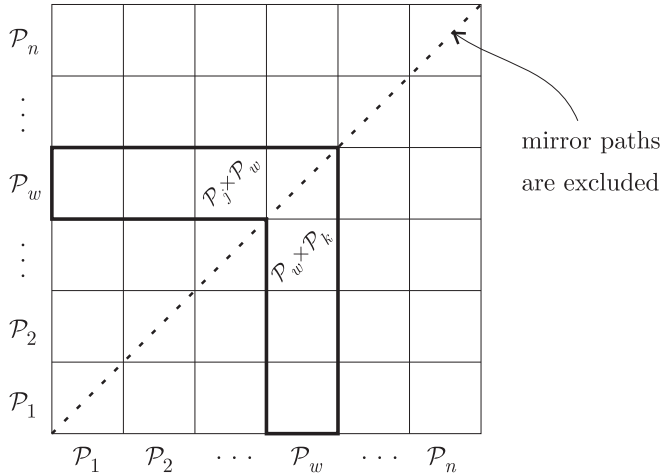


FIG. 6: The set of paths that after  $t$  steps reach the position  $s = s_0$ , with the last coin outcome  $a_t = 1$ , can be expressed in terms of the sets  $\mathcal{P}_w$  as  $[\bigcup_{w,w'} \mathcal{P}_w \times \mathcal{P}_{w'}] \setminus M = [\bigcup_{w=1}^n (\mathcal{P}_w \times \mathcal{P}_w) \cup (\bigcup_{j=1}^{w-1} \mathcal{P}_j \times \mathcal{P}_w) \cup (\bigcup_{k=1}^{w-1} \mathcal{P}_w \times \mathcal{P}_k)] \setminus M$ , where  $M \equiv \{(\vec{a}, \vec{a}') | \vec{a} = \vec{a}'\}$ .

Now we can write an upper bound for  $p_{\text{prop}}(s_0, t)$  as the number of links with even linking numbers divided by the number of all links:

$$p_{\text{prop}}(s_0, t) \leq \frac{2 \left( \sum_{w=1}^n \rho^w \left( |\mathcal{P}_w|^2 - |\mathcal{P}_w| + 2 \sum_{j=1}^{w-1} |\mathcal{P}_w| |\mathcal{P}_j| \right) \right)}{2(\#(all)^2 - \#(all))}. \quad (18)$$

The number 2 in both the nominator and the denominator accounts for the fact that we count both types of paths – with  $a_t = a'_t = 0$  and  $a_t = a'_t = 1$  – and use the symmetry between the two situations. We used the relation (17) and the diagram at Fig.6. Also, we didn't forget to exclude the mirror paths.

Before substituting to (18), we rewrite (15) as

$$|\mathcal{P}_w| = \frac{(2n-1)!}{(n-w-1)!(n+w-1)!} \left( \frac{1}{n-w} - \frac{1}{n+w} \right) = \frac{w}{n} \binom{2n}{n-w} \quad (19)$$

and (13) as

$$\#(all) = \frac{(2n-1)!}{n!(n-1)!} = \frac{1}{2} \binom{2n}{n}. \quad (20)$$

Now (18) reads

$$p_{\text{prop}}(s_0, t) \leq \frac{\frac{1}{2} \binom{2n}{n}}{\frac{1}{2} \binom{2n}{n} - 1} \frac{\sum_{w=1}^n \rho^w \left( \binom{2n}{n-w}^2 - |\mathcal{P}_w| + 2 \sum_{j=1}^{w-1} |\mathcal{P}_w| |\mathcal{P}_j| \right)}{\frac{1}{2} \binom{2n}{n} \frac{1}{2} \binom{2n}{n}},$$

which we estimate from above using the relation between binomial coefficients  $\binom{2n}{n-w} \leq \binom{2n}{n}$  to get:

$$\begin{aligned} p_{\text{prop}}(s_0, t) &\leq \frac{\frac{1}{2} \binom{2n}{n}}{\frac{1}{2} \binom{2n}{n} - 1} \frac{\sum_{w=1}^n \rho^w \left( \frac{w^2}{n^2} + 2 \sum_{j=1}^{w-1} \frac{w}{n} \frac{j}{n} \right)}{\frac{1}{2} \frac{1}{2}} \\ &= \left( 1 + \frac{1}{\frac{1}{2} \binom{2n}{n} - 1} \right) \frac{4}{n^2} \sum_{w=1}^n \rho^w \left( w^2 + 2w \sum_{j=1}^{w-1} j \right) \\ &= \left( 1 + \frac{1}{\frac{1}{2} \binom{2n}{n} - 1} \right) \frac{4}{n^2} \sum_{w=1}^n \rho^w w^3, \end{aligned} \quad (21)$$

where we used the formula  $\sum_{j=1}^{w-1} j = \frac{1}{2} w(w-1)$ . For  $n \geq 2$  (i.e.  $t \geq 4$ )

$$\frac{1}{\frac{1}{2} \binom{2n}{n} - 1} \leq \frac{1}{\frac{1}{2} \binom{4}{2} - 1} = \frac{1}{5}. \quad (22)$$

Thus

$$p_{\text{prop}}(s_0, t) \leq \frac{6}{5} \frac{4}{n^2} \sum_{w=1}^n \rho^w w^3 \leq \frac{6}{5} \frac{4}{n^2} \sum_{w=1}^{\infty} \rho^w w^3. \quad (23)$$

The infinite sum on the right hand side converges for all  $\rho \in [0, 1)$  and one can find that

$$\sum_{w=1}^{\infty} \rho^w w^3 = \frac{\rho(1+4\rho+\rho^2)}{(1-\rho)^4}. \quad (24)$$

Hence we obtain the following upper bound (recall  $n = t/2$ ):

$$p_{\text{prop}}(s_0, t) \leq \frac{1}{t^2} \frac{6 \cdot 4 \cdot 4}{5} \frac{\rho(1+4\rho+\rho^2)}{(1-\rho)^4} \quad (25)$$

$$= \frac{C}{t^2}, \quad (26)$$

where  $C$  doesn't depend on time (number of steps)  $t$ .

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